Geophys. J. R. astr. Soc. (1987) **90**, 75–100

Multiplet–multiplet coupling due to lateral heterogeneity: asymptotic effects on the amplitude and frequency of the Earth's normal modes

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Accepted 1986 November 18. Received 1986 November 10; in original form 1986 May 20

Summary. Starting with the first-order formulation of quasi-degenerate splitting theory for the normal modes of a laterally heterogeneous earth, we have obtained an asymptotic expression for the coupling terms corresponding to neighbouring multiplets along the same dispersion branch as the mode considered, valid to order $1/l$, where l is the angular order of this mode ($l \gg 1$).

We show that, to order zero, these coupling terms introduce a small shift in epicentral distance into the expression for the long period seismogram obtained by normal mode summation. This shift depends on the difference between the great circle and the minor arc averages of the local frequency. The coupling terms thus permit us to reconcile results obtained by normal-mode summation and by a propagating wave approach, as far as the dependence on structure of the phase of surface waves is concerned.

To order $1/l$, the coupling terms result in a perturbation in the amplitude of the mode considered, which depends on spatial derivatives of the local frequency and thus on the structure in the vicinity of the source station great circle path. We show that this term is equivalent to that which is found using ray perturbation methods for propagating surface waves. We compare and discuss the assumptions underlying both approaches and illustrate, by an example, the potential of the asymptotic normal-mode formulation for improved modelling of lateral heterogeneity in the earth.

Key words: normal modes, coupling, asymptotics

Introduction

Measured eigenfrequencies of the Earth's normal modes present shifts with respect to predicted values for a spherically symmetric earth model that are attributed to lateral heterogeneity. They are usually interpreted in the framework of the geometrical optics approximation, in its lowest-order asymptotic expression. According to this theory, they represent the average, over the great circle path containing the source and the station, of the local frequency, a parameter which represents the local properties of the structure at each point on the Earth. The overall validity of this approximation is confirmed by the consistent patterns over the Earth obtained by plotting, for given modes, the measured frequency shift as a function of the pole position of the source-station great circle (Masters *et al.* 1982; Davis 1986).

Several types of observations have recently been accumulating which tend to indicate that there are measurable deviations from this asymptotic theory. One example is the systematic observation of fluctuations in the observed frequency shift of a mode as a function of angular order, as pointed out in the past (Jobert & Roult 1976; Silver & Jordan 1981) and recently documented on the basis of the collection of data from the GEOSCOPE network (Roult, Romanowicz & Jobert 1986; Romanowicz & Roult 1986). Such fluctuations are not accounted for in the lowest-order asymptotic theory classically considered. Another example is the relatively frequent observations of amplitude anomalies on successive long-period surface-wave trains (Lay & Kanamori 1985). The latter are thought to be largely due to deviations from great circle propagation and focusing effects due to lateral heterogeneity. Several techniques have recently been developed to model these effects using propagating surface waves (Yomogida & Aki 1985; Jobert 1986; Woodhouse & Wong 1986).

On the other hand, multiplet frequency shifts are generally viewed in terms of first-order degenerate perturbation theory. To describe them Jordan (1978) introduced the 'location parameter', which is formally expressed in terms of the splitting matrix of the mode considered, and asymptotically yields the great circle average of the local frequency (Jordan 1978; Dahlen 1979). As for the fluctuations in the observed frequency shifts, it has been suggested by Dahlen (1982, unpublished) that they could be explained by higher order effects in the geometrical optics approximation of the location parameter. In a recent paper (Romanowicz & Roult 1986), we derived an asymptotic expression, valid to order $1/l$, where l is the angular order of the mode, for the location parameter, using a stationary phase method and simple geometrical relations on the sphere. The results were favourably compared with observations in an attempt to set up an inversion scheme for a more accurate determination of the great circle average of the local frequency, and to obtain some estimate of its spatial derivatives, which appear in the term of order $1/l$.

The location parameter, however, only depends on the even part of the spherical harmonics expansion of lateral heterogeneity, and is therefore only of limited interest for the study of the Earth's large-scale structure. Moreover, the seismograms obtained by normal-mode summation using first-order degenerate perturbation theory also depend only on these even terms. To remedy this shortcoming, Woodhouse & Dziewonski (1984), in their seismogram calculation by normal-mode summation, introduced a small shift in epicentral distance, which depends on the difference between the great circle and minor arc averages of the local frequency, by an argument of consistency with surface wave results. In fact, the information about the odd part of the asphericity contained in the long-period seismograms is a consequence of coupling between different multiplets (Madariaga & Aki 1972; Luh 1973; Dahlen 1979; Woodhouse 1983). Quasi-degenerate perturbation theory has to be used to include these effects, and its formulation, to first order, has been derived by Woodhouse (1983), who also conjectured that it is the coupling between neighbouring multiplets along a given dispersion branch that yields the main dependence of normal modes on the odd part of the heterogeneity. Although some attention has been given to coupling terms in the literature (Dahlen 1969; Luh 1973, 1974; Park 1986), they are usually neglected in normal mode studies because of the formidable complication represented by the calculation of the corresponding splitting matrix terms.

In this paper, we apply the same asymptotic method to order $1/l$ as developed previously for the location parameter (Romanowicz & Roult 1986, hereafter referred to as Paper I) to the coupling by lateral heterogeneity of neighbouring multiplets belonging to the same dispersion branch. We show how the dependence on the odd part of heterogeneity is introduced both in the phase and in the amplitude of normal modes, how we can reconcile several aspects of normal-mode versus propagating wave approaches, and, in particular, how we can now produce realistic surface-wave amplitude anomalies by normal-mode summation in an aspherical earth.

First order quasi-degenerate perturbation theory

Using quasi-degenerate perturbation theory complete to first order, Woodhouse (1983) derived the following expression for a given component $s(t, \Delta)$ of the acceleration observed at time t and angular distance Δ from the source:

$$s(t, \Delta) = \mathrm{Re}\left(\sum_K A_K \exp(i\omega_k t)\right), \tag{1}$$

where the summation is taken over multiplets K, and attenuation has not been included. Here, ω_k is the eigenfrequency of multiplet K in the reference spherically symmetric earth model and:

$$\begin{aligned} A_K = {} & \sum_m R_K^m S_K^m - \frac{2}{\omega_k}\sum_{mm'} R_K^m H_K^{mm'} S_K^{m'} + it\left(\sum_{mm'} R_K^m (H_K^{mm'} - \omega_k^2 P_K^{mm'}) S_K^{m'}\right) \\ & + \sum_{K' \neq K} \frac{2\omega_k}{(\omega_k^2 - \omega_{k'}^2)}\left(\sum_{mm'} R_K^m (H_{KK'}^{mm'} - \omega_k^2 P_{KK'}^{mm'}) S_{K'}^{m'}\right. \\ & \left. + \sum_{mm'} R_{K'}^{m'} (H_{K'K}^{m'm} - \omega_k^2 P_{K'K}^{m'm}) S_K^m\right), \end{aligned} \tag{2}$$

where $\{H_K\}$ is the splitting matrix for multiplet K, and $\{P_K\}$ is the matrix of perturbations in density. The expressions for $\{H_K\}$ and $\{P_K\}$ in the case of an isolated multiplet can be found in Woodhouse & Dahlen (1978) and Woodhouse & Girnius (1982). In these papers different normalizations of eigenfunctions have been used. We follow here the notations of Woodhouse (1983) in which the vertical eigenfunctions s_k^m corresponding to multiplet K in the reference spherically symmetric model, are normalized according to:

$$\int_v \rho_0(r)\, s_k^m\, s_{k'}^{m'}\, r^2\, dr = \delta_{kk'}\delta_{mm'}, \tag{3}$$

where ρ_0 is the density distribution in the reference model and integration is extended to the volume of the whole earth. With these conventions, $\{H\}$ and $\{Z\}$ have the dimensions of frequency.

Expressions for the coupling terms $Z_{KK'}^{mm'} = H_{KK'}^{mm'} - \omega_k^2 P_{KK'}^{mm'}$ can be found in Woodhouse (1980). R_K^m and S_K^m are, respectively, receiver and source functions, which can be written as follows using an operator formalism, as in Paper I:

$$R_K^m(\theta_R, \phi_R) = \mathbf{Op}_1[Y_l^m(\theta_R, \phi_R)]$$

$$S_K^m(\theta_S, \phi_S) = \mathbf{Op}_2[Y_l^m(\theta_S, \phi_S)]$$

with operators:

$$\mathbf{Op}_1 = (\mathbf{v} \cdot \mathbf{D}), \quad \mathbf{Op}_2 = (\mathbf{M} : \epsilon^*),$$

where **M** is the moment tensor describing the source, **D** is the displacement operator, ϵ the strain operator and **v** the instrument operator.

Equation (1) is often written in the form (Woodhouse 1983; Tanimoto 1984):

$$s(t, \Delta) = \mathrm{Re}\left(\sum_K A'_K \exp(i\bar{\omega}_k t)\right) \tag{4}$$

where $\bar{\omega}_k = \omega_k + \Lambda_K$. Λ_K is the location parameter of Jordan (1978):

$$\Lambda_K = \frac{\sum_{mm'} R_K^m (H_K^{mm'} - \omega_k^2 P_K^{mm'}) S_K^{m'}}{\sum_m R_K^m S_K^m} \tag{5}$$

and

$$A'_K = A_K - it \sum_{mm'} R_K^m (H_K^{mm'} - \omega_k^2 P_K^{mm'}) S_K^{m'} \tag{6}$$

We note here, however, that this formulation is valid only away from the nodes of the radiation pattern, for which the denominator in (4) is close to zero, and implies neglecting coupling terms in the definition of Λ_K.

Woodhouse & Girnius (1982) showed that the splitting matrix $\{H_K\}$ for a single multiplet K could be expressed in terms of three local functionals $\delta\omega_k^i$ of the Earth's structure as follows:

$$H_K^{mm'} = \sum_{i=0}^{i=2} \iint_\Omega \delta\omega_k^i(\theta, \phi) k_i^{mm'}(\theta, \phi)\, d\Omega, \tag{7}$$

where the kernels are:

$$k_i^{mm'}(\theta, \phi) = \frac{(-\nabla_1^2)^i\, Y_l^{m^*}(\theta, \phi)\, Y_{l'}^{m'}(\theta, \phi)}{[2l(l+1)]^i}; \tag{8}$$

here, ∇_1 is the gradient operator on the unit sphere, Y_l^m are fully normalized spherical harmonics, l is the angular order of multiplet K, and it can be shown that $\delta\omega_k^0$ is the local frequency as defined by Jordan (1978). If we now consider the terms of the splitting matrix corresponding to coupling between multiplets K and K', we show in Appendix I that we can similarly define local frequencies $\delta\omega_{kk'}^i$ $(i = 0, 1, 2)$, such that:

$$Z_{KK'}^{mm'} = \sum_{i=0}^{i=2} \iint_\Omega \delta\omega_{kk'}^i(\theta, \phi)\, L_i^{mm'}(\theta, \phi)\, d\Omega \tag{9}$$

with kernels:

$$L_i^{mm'}(\theta, \phi) = \frac{(-\nabla_1^2)^i Y_l^{m^*}(\theta, \phi) Y_{l'}^{m'}(\theta, \phi)}{[2l(l+1)2l'(l'+1)]^{i/2}}, \tag{10}$$

where l and l' are the angular orders of multiplets K and K', respectively. The expressions for $\delta\omega^0_{kk'}$ and $\delta\omega^0_k$ are given in Table 1, for the case of spheroidal modes. We note that if $K = K'$ these expressions are the same.

In this paper, we shall be interested in evaluating expression (2) for the complex amplitude of multiplet K using an asymptotic approach. The first three terms of (2) contain familiar expressions that have already been calculated in Paper I. Let us for the time being concentrate on the calculation of the last term, which contains the contributions of coupling with other multiplets.

Let S_K be the contribution to A_K from coupling with other multiplets:

$$S_K = \sum_{K' \neq K} \frac{2\omega_k}{(\omega_k^2 - \omega_{k'}^2)} P_{K,K'}, \tag{11}$$

with:

$$P_{K,K'} = \sum_{mm'} (R_K^m Z_{KK'}^{mm'} S_{K'}^{m'} + R_{K'}^{m'} Z_{K'K}^{m'm} S_K^m). \tag{12}$$

Using the same operator formalism as in Paper I and as mentioned above, we can transform expression (11), using (9) and (10), to:

$$P_{K,K'} = \sum_{i=0}^{i=2} \iint \mathbf{Op}_i \left(\frac{\sum_{mm'} Y_l^m(\theta_R, \phi_R)\, Y_l^{m*}(\theta, \phi) Y_{l'}^{m'}(\theta, \phi)\, Y_{l'}^{m'*}(\theta_S, \phi_S)}{[2l(l+1)2l'(l'+1)]^{i/2}} \right) \delta\omega^i_{kk'}(\theta, \phi)\, d\Omega$$
$$+ \iint \mathbf{Op}_i \left(\frac{\sum_{mm'} Y_{l'}^{m'}(\theta_R, \phi_R) Y_{l'}^{m'*}(\theta, \phi) Y_l^m(\theta, \phi) Y_l^{m*}(\theta_S, \phi_S)}{[2l(l+1)2l'(l'+1)]^{i/2}} \right) \delta\omega^i_{k'k}(\theta, \phi)\, d\Omega \tag{13}$$

with:

$$\mathbf{Op}_i = \mathbf{Op}_1 (-\nabla_1^2)^i \mathbf{Op}_2.$$

It is possible to move the operators outside the summation on m and m', since they do not depend on the azimuthal order. We also keep in mind that $\mathbf{Op}_1$ acts on the receiver coordinates (θ_R, ϕ_R), $\mathbf{Op}_2$ acts on the source coordinates (θ_S, ϕ_S) and ∇_1 on the running coordinates (θ, ϕ) on the unit sphere Ω.

The addition theorem for spherical harmonics (Edmonds 1960) yields:

$$\sum_m Y_l^m(\theta, \phi) Y_l^{m*}(\theta_S, \phi_S) = \sqrt{\frac{k}{2\pi}}\, Y_l^0(\lambda)$$
$$\sum_{m'} Y_{l'}^{m'*}(\theta, \phi) Y_{l'}^{m'}(\theta_R, \phi_R) = \sqrt{\frac{k'}{2\pi}}\, Y_{l'}^0(\beta) \tag{14}$$

and two other similar expressions with the indexes l and l' interchanged. The angles λ and β are defined in Fig. 1 and $k = l + 1/2, k' = l' + 1/2$.

Then:

$$P_{K,K'} = \sum_{i=0}^{i=2} J^i_{K,K'} \Big/ [2l(l+1)2l'(l'+1)]^{i/2} \tag{15}$$

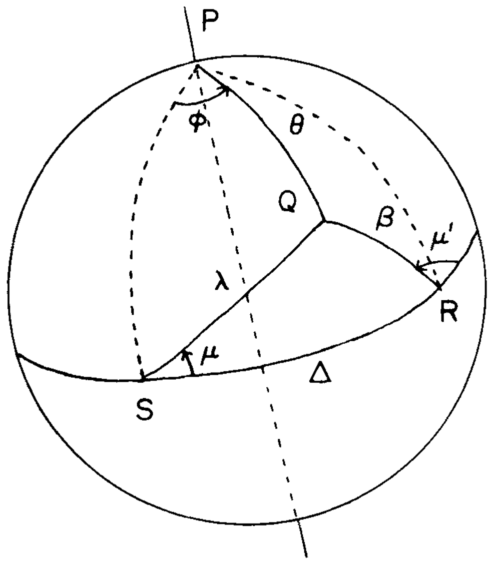

Figure 1. Epicentral coordinate system used in this study. S is the epicentre, R the receiver, Q a point on the surface of the unit sphere, and P the pole of the source receiver great circle γ.

with:

$$J^i_{K,K'} = \sqrt{\frac{k}{2\pi}}\sqrt{\frac{k'}{2\pi}}\left(\iint \delta\omega^i_{kk'}(Q)\,\mathbf{Op}_i[Y^0_{l'}(\lambda)Y^0_l(\beta)]\,d\Omega + \iint \delta\omega^i_{k'k}(Q)\,\mathbf{Op}_i[Y^0_l(\lambda)Y^0_{l'}(\beta)]\,d\Omega\right), \tag{16}$$

where Q is the running point on the sphere (Fig. 1) and the integrals are taken over the unit sphere.

Spheroidal modes on the vertical component: the case of an isotropic source

In this simple case, operators $\mathbf{Op}_1$ and $\mathbf{Op}_2$ amount to multiplications by constant factors, such that:

$$J^i_{KK'} = \sqrt{\frac{k}{2\pi}}\sqrt{\frac{k'}{2\pi}}\,M_0\left(\alpha_{kk'}\iint \delta\omega^i_{kk'}(Q)\,(-\nabla_1^2)^i[Y^0_{l'}(\lambda)Y^0_l(\beta)]\,d\Omega + \alpha_{k'k}\iint \delta\omega^i_{k'k}(Q)\,(-\nabla_1^2)^i[Y^0_l(\lambda)\,Y^0_{l'}(\beta)]\,d\Omega\right), \tag{17}$$

where M_0 is the scalar moment of the source and:

$$\alpha_{kk'} = -[\![\partial_r U(r_s) + 2[U(r_s) - l(l+1)/2\;V(r_s)]/r_s]\!]\,U'(a)$$

$$\alpha_{k'k} = -[\![\partial_r U'(r_s) + 2[U'(r_s) - l'(l'+1)/2\,V'(r_s)]/r_s]\!]\,U(a). \tag{18}$$

Here U, V are the vertical eigenfunctions of the reference model in the notation of Gilbert & Dzienwonski (1975), evaluated at the source ($r = r_s$) and at the receiver ($r = a$). Unprimed and primed functions refer to multiplets K and K' respectively.

RESTRICTION TO A SINGLE DISPERSION BRANCH

In what follows, we shall only consider the contribution to S_K (equation 11) from modes K' belonging to the same dispersion branch as mode K. We shall then set $k' = k + n$, with n an integer, and assume that $|n| \ll l$, so that n/l is of order $1/l$.

This assumption implies that multiplets K and K' are close together so that the radiation pattern and vertical eigenfunctions for multiplet K' in the reference model are only slighly perturbed from those of multiplet K. Recalling that, when $K = K'$:

$$\delta\omega^i_{k,k'} = \delta\omega^i_{k',k} = \delta\omega^i_k;$$

we shall write:

$$\begin{aligned} \alpha_{kk'}\delta\omega^i_{kk'} &= \alpha_{kk}\delta\omega^i_k + \gamma^i_n \\ \alpha_{k'k}\delta\omega^i_{k'k} &= \alpha_{kk}\delta\omega^i_k + \delta^i_n, \end{aligned} \tag{19}$$

where $\delta\omega^i_k$ is the ith local frequency corresponding to multiplet K as defined by Woodhouse & Girnius (1982), and γ^i_n, δ^i_n are small perturbations. We can then write to first-order in the earth model perturbations:

$$J^i_{K,K'} = \sqrt{\frac{k}{2\pi}}\sqrt{\frac{k'}{2\pi}} M_0\, \alpha_{kk} \left(\iint \delta\omega^i_k(Q)\, (-\nabla^2)^i\, [Y^0_{l'}(\lambda)\, Y^0_l(\beta)]\, d\Omega \right.$$
$$\left. + \iint \delta\omega^i_k(Q)\, (-\nabla^2)^i [Y^0_l(\lambda)\, Y^0_{l'}(\beta)]\, d\Omega \right) \tag{20}$$

and we shall neglect in what follows contributions to $J^i_{K,K'}$ coming from the perturbation terms in (19). We shall come back to this approximation later.

We shall now proceed to evaluate the integrals in expressions (20) approximately. For this we shall be using the asymptotic expression for the fully normalized spherical harmonic Y^0_l (Robin 1958), valid to order $1/l$:

$$Y^0_l(x) = \frac{1}{\pi\sqrt{\sin x}} \cos\left(kx - \frac{\pi}{4} - \frac{\cot x}{8k}\right) + O(1/l^2) \tag{21}$$

and we shall evaluate the integrals in (20) approximately to order $1/l$, using, as in Paper I, the stationary phase approximation to that order, whose expression was derived in Appendix I of Paper I and is given here for reference in Appendix II. We shall also be concerned only with the term $J^0_{K,K'}$, as we can show by the same method that the integrals in $J^1_{K,K'}$ and $J^2_{K,K'}$ lead to terms of order $1/l^2$, which we shall neglect. We can then write:

$$J^0_{K,K'} = M_0\, \alpha_{kk}(I_1 + I_2) \tag{22}$$

with:

$$\begin{aligned} I_1 &= \sqrt{\frac{k}{2\pi}}\sqrt{\frac{k'}{2\pi}} \iint \delta\omega^0_k(Q)\, Y^0_{l'}(\lambda)\, Y^0_l(\beta)\, d\Omega \\ I_2 &= \sqrt{\frac{k}{2\pi}}\sqrt{\frac{k'}{2\pi}} \iint \delta\omega^0_k(Q)\, Y^0_l(\lambda)\, Y^0_{l'}(\beta)\, d\Omega. \end{aligned} \tag{23}$$

To evaluate I_1, we shall consider a 'receiver' coordinate system (β, μ'), as defined in Fig. 1, so that, using (21) and expression (II.2) of Appendix II, we obtain:

$$I_1 = \sqrt{\frac{k}{2\pi}}\,\frac{1}{2\pi^2\sqrt{\sin\Delta}} \int_0^{2\pi} \cos\left(k\beta - \frac{\pi}{4} - \frac{\cot\beta}{8k}\right)$$

$$\times \left[\delta\omega_k^0(\beta, 0)\cos\left(k'\lambda_0 - \frac{\pi}{2}\right) - \sin\left(k'\lambda_0 - \frac{\pi}{2}\right)\left(\frac{\delta\omega_k^0}{8k'\lambda_0''} + \frac{1}{2k'\lambda_0''}\,\frac{\partial^2\delta\omega_k^0}{\partial\mu'^2}\right)\right] d\beta, \qquad (24)$$

where:

$$\lambda_0 = \lambda(\beta, 0) = \beta + \Delta; \quad \lambda_0'' = \frac{\partial^2\lambda}{\partial\mu'^2}(\beta, 0) = -\frac{\sin\beta\sin\Delta}{\sin\lambda_0}. \qquad (25)$$

Similarly, I_2 is evaluated using the 'epicentral' coordinate system (λ, μ) defined in Fig. 1, yielding:

$$I_2 = \sqrt{\frac{k}{2\pi}}\,\frac{1}{2\pi^2\sqrt{\sin\Delta}} \int_0^{2\pi} \cos\left(k\lambda - \frac{\pi}{4} - \frac{\cot\lambda}{8k}\right)$$

$$\times \left[\delta\omega_k^0(\lambda, 0)\cos k'\beta_0 - \sin k'\beta_0\left(\frac{\delta\omega_k^0}{8k'\beta_0''} + \frac{1}{2k'\beta_0''}\,\frac{\partial^2\,\delta\omega_k^0}{\partial\mu^2}\right)\right] d\lambda \qquad (26)$$

with:

$$\beta_0 = \beta(\lambda, 0) = \lambda - \Delta; \quad \beta_0'' = \frac{\partial^2\beta}{\partial\mu^2}(\lambda, 0) = \frac{\sin\lambda\sin\Delta}{\sin\beta_0}. \qquad (27)$$

Replacing k' by $k+n$ and expressing, in the case of I_1, λ_0 and λ_0'' in the epicentral coordinate system (λ, μ), we thus obtain after a little algebra:

$$J_{K,K'}^0 = M_0\sqrt{\frac{k}{2\pi}}\,\frac{\alpha_{kk}}{2\pi^2\sqrt{\sin\Delta}}\left[\!\!\left[\int_0^{2\pi}\delta\omega_k^0(\lambda, 0)\right.\right.$$

$$\times\left[\cos\left(k\Delta - \frac{\pi}{4}\right)[\![\cos(n\lambda) + \cos[n(\lambda - \Delta)]]\!]\right.$$

$$\left. + \sin\left(k\Delta - \frac{\pi}{4}\right)[\![\sin[n(\lambda - \Delta)] - \sin(n\lambda)]\!]\right] d\lambda$$

$$+ \int_0^{2\pi} B_k(\lambda, 0)\left[\cos\left(k\Delta - \frac{\pi}{4}\right)[\![-\sin[n(\lambda - \Delta)] + \sin(n\lambda)]\!]\right.$$

$$\left.\left.\left. + \sin\left(k\Delta - \frac{\pi}{4}\right)[\![\cos(n\lambda) + \cos[n(\lambda - \Delta)]]\!]\right] d\lambda\right]\!\!\right] \qquad (28)$$

with:

$$B_k(\lambda, 0) = \frac{\cot\Delta}{8k}\,\delta\omega_k^0(\lambda, 0) + \frac{1}{2k\beta_0''}\,\frac{\partial^2\delta\omega_k^0}{\partial\mu^2}. \qquad (29)$$

The first integral in (28) gives rise to a term of order zero in $1/l$ and the second one to a term of order $1/l$ which contains spatial derivatives of the local frequency.

We shall now proceed to further evaluate the line integrals in (28).

(1) Terms 'of order zero'

Let:

$$I_k^n = \int_0^{2\pi} \delta\omega_k^0(\lambda, 0)\, [\![\cos(n\lambda) + \cos[n(\lambda - \Delta)]]\!]\, d\lambda$$

$$J_k^n = \int_0^{2\pi} \delta\omega_k^0(\lambda, 0)\, [\![\sin[n(\lambda - \Delta)] - \sin(n\lambda)]\!]\, d\lambda. \tag{30}$$

We note that I_k^n is even and J_k^n is odd in n. To order zero in $1/l$, we have from (28) and (30), with $k' = k + n$:

$$J_{K,K'}^0 = M_0 \sqrt{\frac{k}{2\pi}}\, \alpha_{kk} \frac{1}{2\pi^2 \sqrt{\sin\Delta}}\, [I_k^n \cos(k\Delta - \pi/4) + J_k^n \sin(k\Delta - \pi/4)]. \tag{31}$$

We note also that within the approximation considered here:

$$\omega_k^2 - \omega_{k+n}^2 \simeq -2n\omega_k U/a, \tag{32}$$

where a is the earth's radius and $U = a(\partial\omega_k/\partial k)$ is the group velocity at frequency ω_k. From equations (11), (15), (28), (31) and (32), it follows:

$$S_K^0 = M_0 \sqrt{\frac{k}{2\pi}}\, \alpha_{kk} \frac{a}{U} \frac{1}{2\pi^2 \sqrt{\sin\Delta}} \sin(k\Delta - \pi/4) \sum_{n \neq 0} -\frac{J_k^n}{n}, \tag{33}$$

where S_K^0 is the contribution of order zero (in our approximation) to S_K. Now, since for $n \neq 0$;

$$\frac{\sin n\lambda - \sin n(\lambda - \Delta)}{n} = \mathrm{Re} \int_0^{\Delta} \exp[in(\lambda - \phi)]\, d\phi, \tag{34}$$

we have from (30):

$$\frac{J_k^n}{n} = -\mathrm{Re} \int_0^{2\pi} \delta\omega_k^0(\lambda, 0) \int_0^{\Delta} \exp[in(\lambda - \phi)]\, d\phi\, d\lambda \tag{35}$$

$$= -2\pi\, \mathrm{Re} \int_0^{\Delta} \exp(-in\phi)\, d\phi\, (\delta\omega_k^0)^n, \tag{36}$$

where $(\delta\omega_k^0)^n$ is the nth Fourier coefficient in the decomposition of $\delta\omega_k^0(\lambda, 0)$:

$$(\delta\omega_k^0)^n = \frac{1}{2\pi} \int_0^{2\pi} \delta\omega_k^0(\lambda, 0) \exp(in\lambda)\, d\lambda.$$

Finally:

$$\sum_{n \neq 0} \frac{J_k^n}{n} = -2\pi\, \mathrm{Re} \int_0^{\Delta} d\phi \sum_n (\delta\omega_k^0)^n \exp(-in\phi) + 2\pi\, \mathrm{Re} \int_0^{\Delta} d\phi (\delta\omega_k^0)^0$$

$$= 2\pi\Delta(\delta\hat{\omega} - \delta\tilde{\omega}), \tag{37}$$

where we have defined:

$$\delta\hat{\omega} = \frac{1}{2\pi}\int_0^{2\pi} \delta\omega_k^0(s)\,ds$$

$$\delta\tilde{\omega} = \frac{1}{\Delta}\int_0^{\Delta} \delta\omega_k^0(s)\,ds \tag{38}$$

the integrals being taken along the great circle containing source and receiver. We thus obtain:

$$S_K^0 = \sqrt{\frac{k}{2\pi}}\,M_0\,\alpha_{kk}\frac{a\Delta}{U}\frac{\sin(k\Delta - \pi/4)}{\pi\sqrt{\sin\Delta}}(\delta\tilde{\omega} - \delta\hat{\omega}). \tag{39}$$

We note in (37) that if n_0 is the highest order of Fourier components present in $\delta\omega(\lambda, 0)$, then only multiplet interactions with $n < n_0$ should be taken into account. Also, the knowledge of n_0 determines above what frequency this asymptotic theory is valid. The great circle theorem can also be understood from (37), since neglecting coupling terms amounts to considering only the $n = 0$ interaction, therefore only the $n = 0$ Fourier component can be retrieved in that case (R. Snieder, private communication).

(2) Terms 'of order $1/l$'

Let $\delta J_{K,K'}^0$ be the order $1/l$ contribution to $J_{K,K'}^0$. Setting:

$$\delta I_k^n = \int_0^{2\pi} B_k(\lambda, 0)\,[\![\cos[n(\lambda - \Delta)] + \cos(n\lambda)]\!]\,d\lambda$$

$$\delta J_k^n = \int_0^{2\pi} B_k(\lambda, 0)\,[\![\sin[n(\lambda - \Delta)] - \sin(n\lambda)]\!]\,d\lambda, \tag{40}$$

where B_k has been defined in (29), we then have:

$$\delta J_{K,K'}^0 = M_0\sqrt{\frac{k}{2\pi}}\frac{\alpha_{kk}}{2\pi^2\sqrt{\sin\Delta}}[\delta I_k^n \sin(k\Delta - \pi/4) - \delta J_k^n\cos(k\Delta - \pi/4)]. \tag{41}$$

Since δI_k^n is even and δJ_k^n odd in n, using (11), (15), (40) and (41), we obtain:

$$\delta S_K = +\frac{aM_0}{U}\sqrt{\frac{k}{2\pi}}\frac{\alpha_{kk}}{\pi^2\sqrt{\sin\Delta}}\cos(k\Delta - \pi/4)\sum_{n\neq 0}\frac{\delta J_k^n}{n}. \tag{42}$$

We then proceed to calculate the sum in (42), following the same steps as in equations (34)–(37), where we replace $\delta\omega_k^0$ by B_k. We finally obtain:

$$\delta S_K = \frac{M_0\,\alpha_{kk}}{\pi\sqrt{\sin\Delta}}\sqrt{\frac{k}{2\pi}}\cos(k\Delta - \pi/4)\frac{a\Delta}{U}\left(\frac{1}{2k}(\hat{D} - \tilde{D}) + \frac{\cot\Delta}{8k}(\delta\hat{\omega} - \delta\tilde{\omega})\right), \tag{43}$$

where we have defined:

$$D_k = \frac{1}{\beta_0''}\frac{\partial^2\delta\omega_k^0}{\partial\mu^2}(\lambda, 0) = \frac{\sin(\phi - \Delta)}{\sin\Delta}[\partial_\theta^2\delta\omega_k^0\sin\phi - \partial_\phi\delta\omega_k^0\cos\phi] \tag{44}$$

with (θ, ϕ) the coordinates in a polar coordinate system (Fig. 1) and:

$$\tilde{D} = \frac{1}{\Delta} \int_0^{\Delta} D_k(\phi)\, d\phi$$

$$\hat{D} = \frac{1}{2\pi} \int_0^{2\pi} D_k(\phi)\, d\phi. \tag{45}$$

The integrals are taken along the great circle γ containing the source and the receiver. We note that in (44) D_k is the second transverse derivative with respect to the great circle. Expression (43) is therefore valid regardless of the reference frame chosen.

COMPLEX AMPLITUDE OF MULTIPLET K

Let us now go back to expression (2) for the complex amplitude of multiplet K. In this expression, the first term is the amplitude A_K^0 of multiplet K in the reference spherically symmetric model:

$$A_K^0 = \sum_m R_K^m S_K^m = M_0 \alpha_{kk} \sqrt{\frac{k}{2\pi}}\, Y_l^0(\Delta). \tag{46}$$

Using the same order of approximation for the spherical harmonic in (46), this yields:

$$A_K^0 = \frac{M_0 \alpha_{kk}}{\pi \sqrt{\sin \Delta}} \sqrt{\frac{k}{2\pi}} \cos\left(k\Delta - \frac{\pi}{4} - \frac{\cot \Delta}{8k}\right). \tag{47}$$

The second term in (2) is:

$$A_K^1 = -\frac{2}{\omega_k} \sum_{mm'} R_K^m H_K^{mm'} S_K^{m'} = -\frac{2}{\omega_k} \mathrm{NUM}(\Lambda_K), \tag{48}$$

where $\mathrm{NUM}(\Lambda_K)$ is the numerator in the expression for the location parameter Λ_K, whose expression to order $(1/l)$ was obtained in Paper I (equation 15):

$$\mathrm{NUM}(\Lambda_K) = M_0 \alpha_{kk} \frac{1}{\pi \sqrt{\sin \Delta}} \sqrt{\frac{k}{2\pi}} \left(\delta\omega \cos(k\Delta - \pi/4 - \cot \Delta/8k) + \frac{\hat{D}}{2k} \sin(k\Delta - \pi/4)\right), \tag{49}$$

where we have assumed that $H_K^{mm'} \simeq H_K^{mm'} - \omega_k^2 P_K^{mm'}$, which is justified within the approximation considered.

The contribution S_K from multiplets $K' \neq K$ along the same dispersion branch as K is, from (39) and (43):

$$S_K = M_0 \alpha_{kk} \frac{1}{\pi \sqrt{\sin \Delta}} \sqrt{\frac{k}{2\pi} \frac{a\Delta}{U}} \left[\sin(k\Delta - \pi/4)(\delta\tilde{\omega} - \delta\hat{\omega}) + \cos(k\Delta - \pi/4) \left(\frac{1}{2k}(\hat{D} - \tilde{D}) + \frac{\cot \Delta}{8k}(\delta\hat{\omega} - \delta\tilde{\omega})\right)\right]. \tag{50}$$

In practice, the term A_K^1 is of order $1/\omega_k t$ with respect to S_K so that we can neglect it, since we are considering, in general, times much larger than one period.

The third term in equation (2) is again the numerator of the location parameter Λ_K multiplied by (it).

Finally, we can write the complex amplitude of multiplet K as follows:

$$A_K \simeq A_K^0 + S_K + it\,[\mathrm{NUM}(\Lambda_K)] \tag{51}$$

or, using approximate expressions up to order $1/l$ (equations 46, 49, 50):

$$\begin{aligned} A_K = \frac{M_0 \alpha_{kk}}{\pi\sqrt{\sin\Delta}}\sqrt{\frac{k}{2\pi}}\Bigg[&\cos(k\Delta - \pi/4 - \cot\Delta/8k) + \frac{a\Delta}{U}(\delta\bar{\omega} - \delta\hat{\omega})\sin(k\Delta - \pi/4) \\ &+ \cos(k\Delta - \pi/4)\frac{a\Delta}{U}\left(\frac{1}{2k}(\hat{D} - \bar{D}) + \frac{\cot\Delta}{8k}(\delta\hat{\omega} - \delta\bar{\omega})\right) \\ &+ it[\delta\hat{\omega}\cos(k\Delta - \pi/4 - \cot\Delta/8k)] + \frac{\hat{D}}{2k}\sin(k\Delta - \pi/4)\Bigg] \end{aligned} \tag{52}$$

and we note that we can write, to order $1/l$:

$$\mathrm{Re}\,(A_K) \simeq A_K^0(\Delta + \delta\Delta)\,[1 + \delta F(\Delta)], \tag{53}$$

where we have defined a shift in epicentral distance:

$$\delta\Delta = \frac{a\Delta}{kU}(\delta\hat{\omega} - \delta\bar{\omega}) \tag{54}$$

and an amplitude perturbation:

$$\delta F(\Delta) = \frac{a\Delta}{U}\left(\frac{\hat{D} - \bar{D}}{2k} + \frac{\cot\Delta}{8k}(\delta\hat{\omega} - \delta\bar{\omega})\right). \tag{55}$$

We note that $\delta\Delta$ is the same shift in epicentral distance as introduced by Woodhouse & Dziewonski (1984) to account for the odd part of lateral heterogeneity in their synthetic seismogram calculation by normal mode summation. The synthetic seismogram obtained here can also be written, away from the zeroes of $A_K^0(\Delta + \delta\Delta)$, in the form:

$$s(t, \Delta) = \sum_K [\![A_K^0(\Delta + \delta\Delta)\,[1 + \delta F(\Delta)]\,\exp\,[it(\omega_k + \Lambda_k)]]\!], \tag{56}$$

where Λ_K is the multiplet frequency shift. As we shall see below, this expression differs from that of Woodhouse & Dziewonski (1984) essentially by the amplitude perturbation term $\delta F(\Delta)$, and it is this term $\delta F(\Delta)$ that governs the amplitude variations of the propagating surface waves. In order to avoid problems near the nodes of the radiation pattern, we shall however prefer to write $s(t, \Delta)$ in the form, valid at all epicentral distances away from the poles ($\Delta = 0, \pi$):

$$s(t, \Delta) = \mathrm{Re}\left[\!\!\left[\sum_k a_k\,[G_1(\Delta)\cos(k\Delta - \pi/4) + G_2(\Delta)\sin(k\Delta - \pi/4)]\,\exp\,(i\omega_k t)\right]\!\!\right], \tag{57}$$

where:

$$a_k = M_0 \frac{\alpha_{kk}}{\pi\sqrt{\sin\Delta}} \sqrt{\frac{k}{2\pi}}$$

$$G_1(\Delta) = 1 + \frac{a\Delta}{U}\left[\left(\frac{\hat{D}-\ddot{D}}{2k}\right) + \frac{\cot\Delta}{8k}(\delta\hat{\omega} - \delta\ddot{\omega})\right] + it\,\delta\hat{\omega}$$

$$G_2(\Delta) = \frac{\cot\Delta}{8k} - \frac{a\Delta}{U}(\delta\hat{\omega} - \delta\ddot{\omega}) + it\left(\frac{\hat{D}}{2k} + \frac{\cot\Delta}{8k}\delta\hat{\omega}\right). \tag{58}$$

Before discussing the consequences of these results, we shall present the corresponding derivation in the more general case of a source represented by a moment tensor $\{M_{ij}\}$.

General case: source represented by a moment tensor $\{M_{ij}\}$

Going back to equations (13) and (16), we have in this case:

$$\mathbf{Op}_1[Y_l^0(\lambda)\,Y_{l'}^0(\beta)] = [a_0X_l^0(\lambda) + a_1X_l^1(\lambda) + a_2X_l^2(\lambda)]\,Y_{l'}^0(\beta)\,U'(a)$$

$$\mathbf{Op}_1[Y_{l'}^0(\lambda)\,Y_l^0(\beta)] = [a_0'X_{l'}^0(\lambda) + a_1'X_{l'}^1(\lambda) + a_2'X_{l'}^2(\lambda)]\,Y_l^0(\beta)\,U(a) \tag{59}$$

with, using the notations of Gilbert & Dziewonski (1975):

$$a_0 = -[M_{rr}\partial_r U + (M_{\theta\theta} + M_{\phi\phi})(U - f^2V/2)r^{-1}]$$

$$a_1 = +\sqrt{l(l+1)}\,(M_{r\theta}\cos\phi + M_{r\phi}\sin\phi)\left(\partial_r V + \frac{U-V}{r}\right)$$

$$a_2 = -l(l+1)\frac{V}{r}[(M_{\theta\theta} - M_{\phi\phi})\cos 2\phi + 2M_{\theta\phi}\sin 2\phi]/2, \tag{60}$$

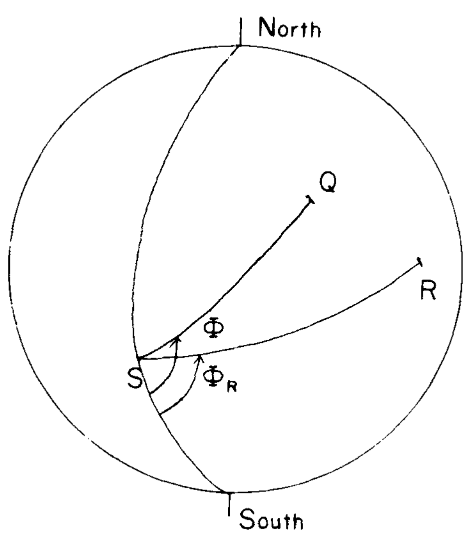

Figure 2. Definition of angles Φ and Φ_R as used in text.

where $f = \sqrt{k/2\pi}$, $\phi = \phi_R + \mu$, the angle ϕ_R is defined in Fig. 2, and the eigenfunctions U, V and their derivatives are evaluated at the source depth. We have similar expressions for a'_i, $i = 0, 1, 2$, replacing l by l'.

As in the case of an isotropic source, we shall restrict ourselves to multiplets belonging to the same dispersion branch, with $n \ll l$, and replace primed functions in (59) by unprimed ones, $\delta\omega^0_{kk'}$ and $\delta\omega^0_{k'k}$ by $\delta\omega^0_k$ and consider only $J^i_{K,K'}$ for $i = 0$. Following the steps of Paper I, we can write a_1 and a_2 in the form:

$$a_1(\phi) = A_1 \cos\mu + B_1 \sin\mu$$

$$a_2(\phi) = A_2 \cos 2\mu + B_2 \sin 2\mu \tag{61}$$

with:

$$A_1 = a_1(\phi_R)$$

$$A_2 = a_2(\phi_R)$$

$$B_1 = + \sqrt{l(l+1)}\,(-M_{r\theta} \sin\phi_R + M_{r\phi} \cos\phi_R)\,[\partial_r V + (U - V)/r]$$

$$B_2 = -V/r\, l(l+1)\,[-(M_{\theta\theta} - M_{\phi\phi}) \sin 2\phi_R + 2M_{\theta\phi} \cos 2\phi_R]/2 \tag{62}$$

so that:

$$J^0_{K,K'} = \sqrt{\frac{k}{2\pi}}\sqrt{\frac{k'}{2\pi}}\,(a_0 Z_0 + A_1 Z_1 + A_2 Z_2 + B_1 T_1 + B_2 T_2), \tag{63}$$

where:

$$Z_0 = \iint \delta\omega^0_k [X^0_l(\lambda)\, X^0_{l'}(\beta) + X^0_{l'}(\lambda)\, X^0_l(\beta)]\, d\Omega$$

$$Z_1 = \iint \delta\omega^0_k [X^1_l(\lambda)\, X^0_{l'}(\beta) + X^1_{l'}(\lambda)\, X^0_l(\beta)] \cos\mu\, d\Omega$$

$$Z_2 = \iint \delta\omega^0_k [X^2_l(\lambda)\, X^0_{l'}(\beta) + X^2_{l'}(\lambda)\, X^0_l(\beta)] \cos 2\mu\, d\Omega$$

$$T_1 = \iint \delta\omega^0_k [X^1_l(\lambda)\, X^0_{l'}(\beta) + X^1_{l'}(\lambda)\, X^0_l(\beta)] \sin\mu\, d\Omega$$

$$T_2 = \iint \delta\omega^0_k [X^2_l(\lambda)\, X^0_{l'}(\beta) + X^2_{l'}(\lambda)\, X^0_l(\beta)] \sin 2\mu\, d\Omega. \tag{64}$$

We note that Z_0 has been evaluated previously, when dealing with the case of the isotropic source. Similarly, we can apply the stationary phase method to evaluate the other four integrals, making use of the asymptotic expansion (paper I) of the normalized Legendre function X^m_l to order $1/l$. In each case, we obtain an expression of similar form as equation (28), from which we infer, after some algebra,

To order zero:

$$S^0_K = \sqrt{\frac{k}{2\pi}}\,\frac{a\Delta}{U}\,\frac{1}{\pi\sqrt{\sin\Delta}}\,[Q_1 \cos(k\Delta - \pi/4) + Q_2 \sin(k\Delta - \pi/4)] \tag{65}$$

with:

$$Q_1 = (\delta\bar{\omega} - \delta\hat{\omega})A_1$$

$$Q_2 = (\delta\bar{\omega} - \delta\hat{\omega})(a_0 - A_2)$$

We verify that, as for the case of the isotropic source, the perturbation in the real part of the complex amplitude due to (65) amounts to a shift $\delta\Delta$ in epicentral distance, as defined in (54).

To order $(1/l)$, we obtain:

$$\delta S_K = \sqrt{\frac{k}{2\pi}}\,\frac{a\Delta}{U}\,\frac{1}{\pi\sqrt{\sin\Delta}}\,[\delta Q_1 \cos(k\Delta - \pi/4) + \delta Q_2 \sin(k\Delta - \pi/4)] \tag{66}$$

with:

$$\delta Q_1 = (a_0 - A_2)\left(\frac{1}{2k}(\hat{D} - \bar{D}) + \frac{\cot\Delta}{8k}(\delta\hat{\omega} - \delta\bar{\omega})\right)$$

$$+ 2\frac{A_2 \cot\Delta}{k}(\delta\hat{\omega} - \delta\bar{\omega}) - 2\frac{B_2}{k}(\hat{E} - \bar{E})$$

$$\delta Q_2 = -A_1\left(\frac{1}{2k}(\hat{D} - \bar{D}) + \frac{\cot\Delta}{8k}(\delta\hat{\omega} - \delta\bar{\omega})\right) + A_1\frac{\cot\Delta}{2k}(\delta\hat{\omega} - \delta\bar{\omega}) - \frac{B_1}{k}(\hat{E} - \bar{E}), \tag{67}$$

where we have defined:

$$\bar{E} = \frac{1}{\Delta}\int_0^{\Delta} \frac{1}{\beta''_0}\frac{\partial\delta\omega_k^0(\lambda,0)}{\partial\mu}\,d\lambda = \frac{1}{\Delta\sin\Delta}\int_0^{\Delta} \sin(\Delta - \phi)\,\partial_\theta\delta\omega_k^0(\phi)\,d\phi$$

$$\hat{E} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\beta''_0}\frac{\delta\omega_k^0(\lambda,0)}{\delta\mu}\,d\lambda = \frac{1}{2\pi\sin\Delta}\int_0^{2\pi} \sin(\Delta - \phi)\,\partial_\theta\delta\omega_k^0(\phi)\,d\phi. \tag{68}$$

We note that the integrands on the left-hand side of (68) are the transverse gradients with respect to the great circle. Expressions (66) and (67) are therefore valid regardless of the reference frame. Recalling the order $1/l$ expression given in equation (25) of Paper I:

$$\sum_{mm'} R_K^m(H_K^{mm'} - \omega_k^2 P_K^{mm'})S_K^{m'} = Q_3\cos(k\Delta - \pi/4) + Q_4 \sin(k\Delta - \pi/4), \tag{69}$$

with:

$$Q_3 = (a_0 - A_2)\,\delta\hat{\omega} + A_1\frac{\hat{D}}{2k} + \left(\frac{1}{8k} - \frac{1}{2k}\right)A_1\cot\Delta\,\delta\hat{\omega} + B_1\frac{\hat{E}}{k}$$

$$Q_4 = -A_1\delta\hat{\omega} + (a_0 - A_2)\frac{\hat{D}}{2k} + \frac{a_0}{8k}\cot\Delta\,\delta\hat{\omega} + \left(\frac{2}{k} - \frac{1}{8k}\right)A_2\delta\hat{\omega} - 2B_2\frac{\hat{E}}{k} \tag{70}$$

and since:

$$\sum R_K^m S_K^m = \sqrt{\frac{k}{2\pi}}\,\frac{1}{\pi\sqrt{\sin\Delta}}\,(a_0X_l^0 + A_1X_l^1 + A_2X_l^2) \tag{71}$$

we obtain, to order $1/l$, using (2), (65) to (71) and the order $1/l$ approximation to the Legendre functions:

$$s(t,\Delta) = \frac{1}{\pi\sqrt{\sin\Delta}}\,\mathrm{Re}\left(\sum_K \sqrt{\frac{k}{2\pi}}\,[G_1(\Delta)\cos(k\Delta - \pi/4)\right.$$

$$\left. + G_2(\Delta)\sin(k\Delta - \pi/4)]\,\exp(i\omega_k t)\right) \tag{72}$$

with:

$$G_1(\Delta) = (a_0 - A_2) - A_1 \left(\frac{1}{2k} - \frac{1}{8k}\right) \cot\Delta + \frac{a\Delta}{U}(Q_1 + \delta Q_1) + itQ_3$$

$$G_2(\Delta) = -A_1 + \frac{a_0 \cot\Delta}{8k} + A_2 \left(\frac{2}{k} - \frac{1}{8k}\right) \cot\Delta + \frac{a\Delta}{U}(Q_2 + \delta Q_2) + itQ_4 \tag{73}$$

We verify that, in the case of an isotropic source ($A_1 = A_2 = B_1 = B_2 = 0$), expressions (72) and (73) reduce to (57) and (58), respectively.

Consequences for normal modes and propagating surface waves

1 EXPRESSION FOR THE MULTIPLET FREQUENCY SHIFT

In equations (72) and (73), G_1 and G_2 have the form:

$$G_1 = \alpha_1 + it\beta_1$$

$$G_2 = \alpha_2 + it\beta_2. \tag{74}$$

According to equation (4), $s(t, \Delta)$ can be written, for short enough times, and away from the nodes:

$$s(t, \Delta) = \frac{1}{\pi\sqrt{\sin\Delta}} \operatorname{Re}\left(\sum_K \sqrt{\frac{k}{2\pi}} [\alpha_1 \cos(k\Delta - \pi/4) + \alpha_2 \sin(k\Delta - \pi/4)] \exp[i(\omega_k + \Lambda_k)t]\right), \tag{75}$$

where the multiplet frequency-shift with respect to the reference spherically symmetric earth model is now:

$$\Lambda_K = \frac{\beta_1 \cos(k\Delta - \pi/4) + \beta_2 \sin(k\Delta - \pi/4)}{\alpha_1 \cos(k\Delta - \pi/4) + \alpha_2 \sin(k\Delta - \pi/4)}. \tag{76}$$

We first note that, to zeroth order in $(1/l)$ and in the model perturbations, this expression comes down to:

$$\Lambda_k^0 = \delta\hat{\omega} \tag{77}$$

which is the familiar result of the lowest order expression of the geometrical optics limit (Jordan 1978; Dahlen 1979).

If we now go back to expression (75) and consider terms up to order $(1/l)$, we note that we can write, away from the nodes of the radiation pattern:

$$\Lambda_K = \delta\hat{\omega} + \left(\frac{\hat{D}}{2k} + x\frac{\hat{E}}{k}\right) \tan(k\Delta - \pi/4 + \chi) \tag{78}$$

with:

$$\chi = \frac{2B_2(a_0 - A_2) + A_1 B_1}{(a_0 - A_2)^2 + A_1^2},$$

a result obtained previously using degenerate perturbation theory and asymptotic expressions for the location parameter (Romanowicz & Roult 1986). To order $(1/l)$ and to first order in the model perturbations, the multiplet frequency shift depends only on great circle averages of the local frequency and its spatial derivatives, and not on the minor arc averages.

2 CONTRIBUTION TO THE PHASE OF SURFACE WAVES

For simplicity, the following derivation will be presented for the case of an isotropic source, but it can readily be extended to the more general case of a source represented by a moment tensor. Going back to expression (57) and the corresponding definitions (58), we can express in them $\cos(k\Delta - \pi/4)$ and $\sin(k\Delta - \pi/4)$ in terms of exponentials, so that we obtain:

$$s(t, \Delta) = \mathrm{Re}\left[\sum_K a_k \exp(i\omega_k t)\left(\frac{G_1(\Delta) - iG_2(\Delta)}{2}\exp[i(k\Delta - \pi/4)]\right.\right.$$

$$\left.\left.+\frac{G_1(\Delta) + iG_2(\Delta)}{2}\exp[-i(k\Delta - \pi/4)]\right)\right]. \tag{79}$$

Transforming the discrete sum over k into an integral over the continuous variable ν, such that $\nu = k$ when l is an integer, using for example Watson's transformation (Aki & Richards 1980) and the residue theorem, we shall obtain two families of terms whose phases are of the form:

$$\Psi^+ = \omega t - k\Delta - \pi/4 + \dots$$

$$\Psi^- = \omega t + k\Delta + \pi/4 + \dots .$$

The first family of terms will give rise to odd-order trains of surface waves ($R_1, R_3 \dots$) and the second one, to even-order trains ($R_2, R_4 \dots$). We shall only be interested here in the first term from which we shall extract the first Rayleigh wavetrain R_1. To obtain its amplitude and phase, we need to consider the $\exp[-i(k\Delta + \pi/4)]$ term in equation (79):

$$R_1(t, \Delta) = \mathrm{Re}\sum_K a_k/2 \exp[i(\omega_k t - k\Delta - \pi/4)]\,[G_1(\Delta) + iG_2(\Delta)]. \tag{80}$$

When isolating the first arriving train, we can replace in expressions (58) the time t by $a\Delta/U$, this substitution being valid to order zero in $1/l$. We then obtain:

$$G_1(\Delta) + iG_2(\Delta) = 1 - \frac{a\Delta}{U}\left(\frac{\tilde{D}}{2k} + \frac{\cot\Delta}{8k}\delta\bar{\omega}\right) + i\left(\frac{a\Delta}{U}\delta\bar{\omega} + \frac{\cot\Delta}{8k}\right). \tag{81}$$

The phase of the first surface wave train will therefore be:

$$\Psi = \omega t - k\Delta - \pi/4 + \frac{a\Delta}{U}\delta\bar{\omega} + \frac{\cot\Delta}{8k}, \tag{82}$$

but:

$$\frac{a\Delta}{U}\delta\bar{\omega} = \frac{\omega}{C_0^2}\int_0^{a\Delta}\delta C(x)\,dx, \tag{83}$$

where C_0 is the phase velocity in the reference spherically symmetric earth model, δC is the perturbation in phase velocity, and the integral is taken along the great circle γ. Hence:

$$\Psi = \omega t - \omega \int_0^{a\Delta} \frac{dx}{C} - \pi/4 + \frac{\cot\Delta}{8k}, \tag{84}$$

where C is the phase velocity in the perturbed model. We obtain in this way, the correct expression for the phase of the first Rayleigh wave train, which, in particular, takes into account the variation of structure along the portion of path between the source and the receiver. We therefore see that, as conjectured by Dahlen (1979), we need to include the contribution of multiplet–multiplet interaction in order to reconcile the normal-mode and propagating-wave approaches in the lowest order approximation of the geometrical optics limit. At the same time, the perturbation $\delta\Delta$ explicitly introduces into the synthetic seismogram the effect of the odd part of the lateral heterogeneity.

We note that the term $\cot\Delta/8k$ in (84) comes from the order $(1/l)$ approximation to the Legendre function and is the same correction as introduced by Wielandt (1980) in his study of the polar phase shift. In the more general case of the moment tensor source, the phase Ψ differs from (84) by an additional source phase.

3 AMPLITUDE PERTURBATION IN THE PROPAGATING SURFACE WAVES

Let us write in equations (58) and (73):

$$G_1(\Delta) = G_1^0(\Delta) + \delta G_1$$

$$G_2(\Delta) = G_2^0(\Delta) + \delta G_2, \tag{85}$$

where G_1^0 and G_2^0 are the corresponding expressions for the reference spherically symmetric earth:

$$G_1^0(\Delta) = (a_0 - A_2) - A_1\left(\frac{1}{2k} - \frac{1}{8k}\right)\cot\Delta$$

$$G_2^0(\Delta) = -A_1 + a_0\frac{\cot\Delta}{8k} + A_2\cot\Delta\left(\frac{2}{k} - \frac{1}{8k}\right). \tag{86}$$

Then, the relative perturbation in amplitude for the first arriving surface wavetrain will be, to first order, using (79) and (86):

$$\frac{\delta A}{A} = \frac{G_1^0\delta G_1 + G_2^0\delta G_2}{(G_1^0)^2 + (G_2^0)^2}. \tag{87}$$

In the case of an isotropic source, for which $A_1 = A_2 = 0$, and using (58), this yields:

$$\frac{\delta A}{A} = -\frac{a\Delta}{U}\left(\frac{\tilde{D}}{2k}\right) \tag{88}$$

which can be written in terms of perturbation in the phase velocity (at constant frequency) instead of the perturbation in local frequency (at constant k), as follows:

$$\frac{\delta A}{A} = \left(\frac{1}{2\sin\Delta}\right)\int_0^{\Delta}\left[\!\left[\sin(\Delta-\phi)\left[\partial_\theta^2\left(\frac{\delta C}{C_0}\right)\right]\sin\phi - \partial_\phi\left(\frac{\delta C}{C_0}\right)\cos\phi\right]\!\right] d\phi. \tag{89}$$

The expression in (89) is the same as obtained by Woodhouse & Wong (1986) for the perturbation in amplitude for a surface wavetrain using a ray perturbation approach.

In the more general case of a moment tensor source, there is an additional term involved due to the radiation pattern of the source, and:

$$\frac{\delta A}{A} = -\frac{a\Delta}{U}\left[\frac{\ddot{D}}{2k} + \frac{\ddot{E}}{k}\left(\frac{2B_2(a_0 - A_2) + A_1B_1}{(a_0 - A_2)^2 + A_1^2}\right)\right]. \tag{90}$$

As an illustration to the preceding calculations, we present, in Fig. 3, the synthetic seismogram calculations showing how we can now produce amplitude anomalies by normal-mode summation using this asymptotic formulation. The example is taken from the case of the Akita Oki event of 1983 May 26 observed at the GEOSCOPE station PAF, for which the source–station great circle path goes through regions of high lateral-gradients in model M84C. The traces are all normalized to the maximum amplitude in the R_1 train and the seismograms are calculated using the source parameters of the centroid solution given in the PDE bulletins. In all cases, an attenuation factor has been included, according to model

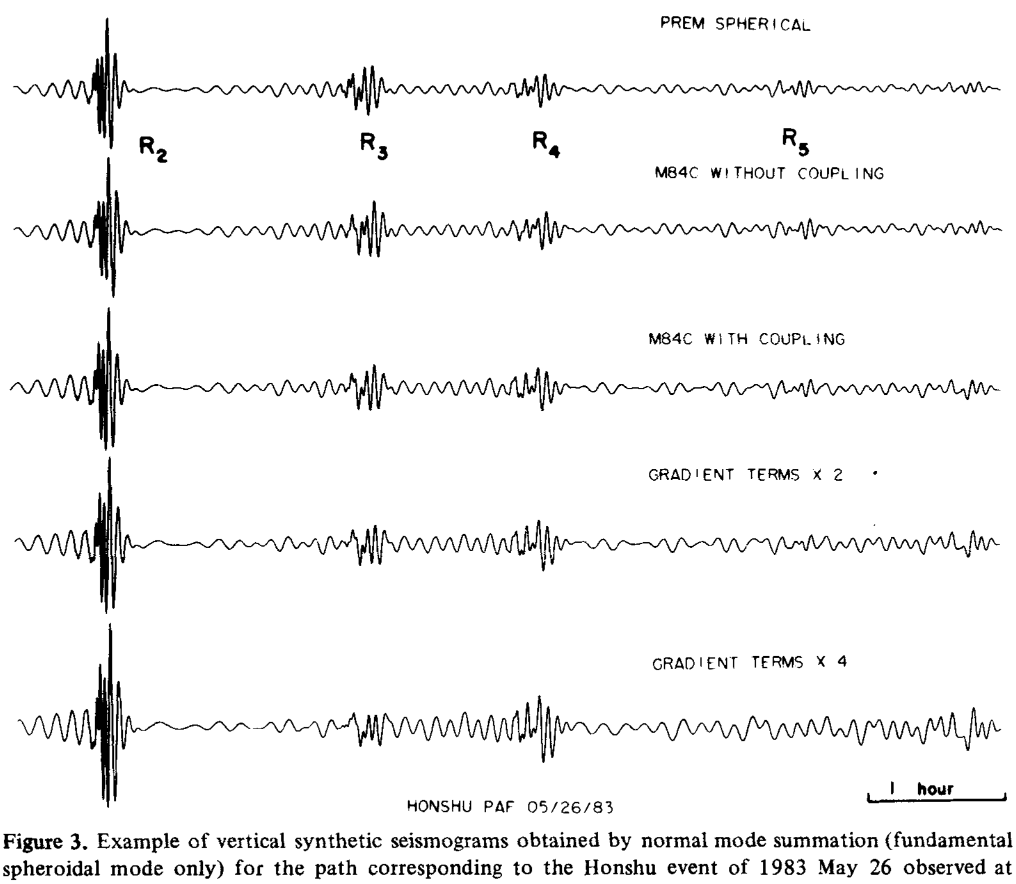

Figure 3. Example of vertical synthetic seismograms obtained by normal mode summation (fundamental spheroidal mode only) for the path corresponding to the Honshu event of 1983 May 26 observed at GEOSCOPE station PAF. The source mechanism is the centroid solution of the PDE for this event and different earth models and formulations are used: (1), (2) – without including coupling terms; (3), (4), (5) – with coupling terms. In (4) and (5), spatial derivatives of the local frequency have been multiplied by a factor with respect to model M84C. All traces are normalized to the maximum amplitude in R_1.

PREM (Dziewonski & Anderson 1981). The top trace is the synthetic seismogram obtained by mode summation (fundamental mode only) for the case of the spherically symmetric earth model PREM. We note the steady decrease in amplitude from one train to the next due to physical dispersion and attenuation. The second trace is the synthetic seismogram obtained for model M84C of Woodhouse & Dziewonski (1984) but without including terms due to coupling. We note some effect on the phase in trains R_3 and R_4 but the amplitude envelope has not changed significantly as compared with trace 1. The third trace is obtained using M84C, including coupling terms with the formulation of equations (79) and (80). We note that the amplitude of R_4 is now as large as that of R_3 and R_5 tends to disappear. This effect is increased as we modify the earth model (somewhat arbitrarily) by multiplying the 'gradient' terms $\hat{D}$, $\bar{D}$, $\hat{E}$, $\bar{E}$ by two (trace 3) and by four (trace 4) with respect to the actual model M84C. Such an effect could not be obtained by using solely the formulation of Woodhouse & Dziewonski (1984), which yields only perturbations in the phase of the surface waves.

Finally, in Fig. 4 we show an example of successful qualitative agreement of observed amplitude anomalies with synthetic ones. This example corresponds to the Chagos Islands event of 1983 November 30 observed at GEOSCOPE station TAM (Algeria). The top trace is the observed vertical seismogram, to which variable filtering has been applied in order to isolate the fundamental mode. A distinct amplitude anomaly is observed when comparing

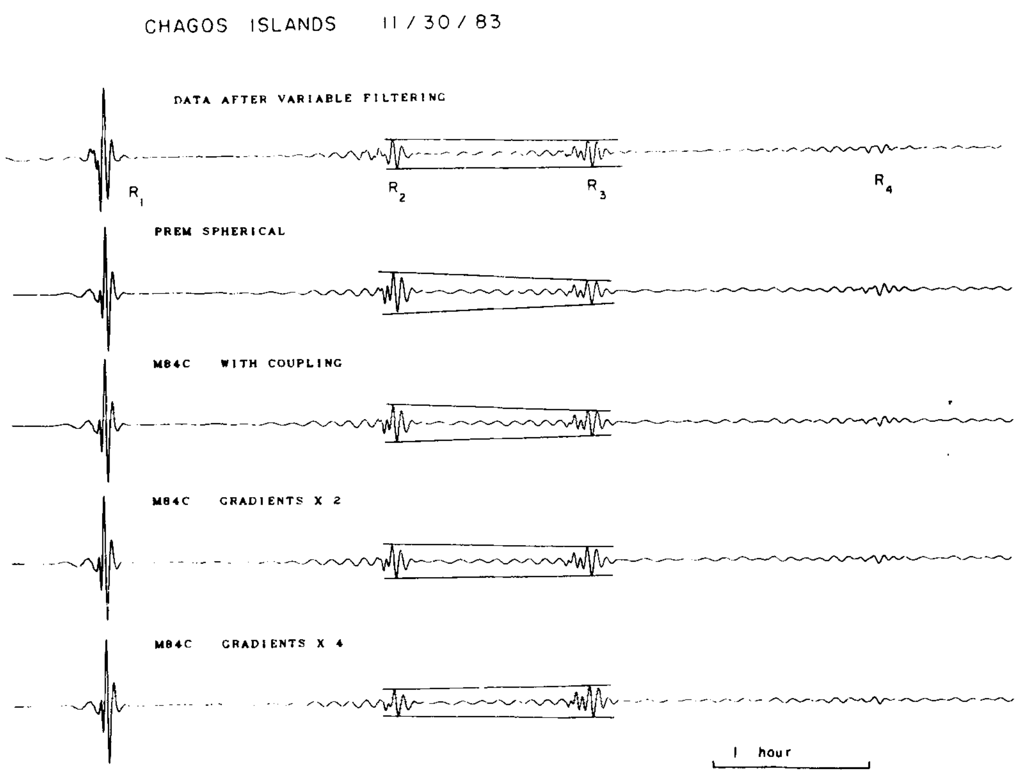

Figure 4. Example of observed vertical seismogram, corresponding to the Chagos Island event of 1986 November 30 observed at GEOSCOPE station TAM and compared to synthetics obtained by normal mode summation, as in Fig. 3. The fundamental spheroidal mode has been isolated on the observed trace by variable filtering. The source mechanism used in the synthetics is the CMTS solution of the PDE bulletin.

trains R_2 and R_3. Again, we present synthetic seismograms successively for a spherical earth model, Model M84C without coupling terms and with coupling terms, and finally a model derived from M84C by multiplying gradients by a factor 2 (and also 4) and taking coupling terms into account. We begin to see an amplitude anomaly for model M84C with coupling terms, and obtain reasonably good visual agreement with the data, when the gradients are multiplied by two. This is only a qualitative experiment, but suggests that amplitude anomalies can perhaps successfully be modelled using this approach. It also suggests that present global models underestimate gradients in lateral heterogeneity, which is not surprising, because of their smooth characteristics.

Discussion

We have calculated the contribution to normal mode amplitude and phase of coupling terms between neighbouring multiplets along a given dispersion branch, by an approximate method. This has enabled us to introduce effects of the odd part of lateral heterogeneity into the long-period seismograms calculated by normal-mode summation, and to reconcile, to the lowest order, the normal-mode approach with the propagating wave formulation in which the phase of successive wavetrains depends on the average phase velocity on the portion of great circle between the epicentre and the receiver. Without the coupling terms, the phase of surface waves obtained by normal-mode summation depended only on the great circle average phase velocity, an obviously incorrect result (Dahlen 1979).

The approximation used involves asymptotic developments in terms of $(1/l)$ of Legendre functions and integrals over the sphere calculated by the stationary phase method. We have seen that, to order zero in $(1/l)$, only a phase shift, as just described, is introduced. The formulation is then equivalent to that of Woodhouse & Dziewonski (1984), who guessed at the correct phase shift to reconcile normal mode and surface wave results. We note, however, that this formulation is valid only in the lowest-order asymptotic approximation to geometrical optics. Moreover, to recover amplitude perturbations such as those due to focusing effects along the source station path, we need to extend the asymptotic calculations up to order $1/l$. When we do so, we introduce an amplitude perturbation which is equivalent, to first order in the model perturbations, to that obtained using ray theory and a propagating wave approach (Woodhouse & Wong 1986). The focusing and defocusing of rays which produces these amplitude variations in the ray theoretical approach are here expressed through the dependence of normal-mode amplitude on the spatial derivatives of the local frequency, along the great circle, that is, on the structure in the vicinity of the great circle.

With the approach considered here, we can now model effects of both even and odd lateral heterogeneity on phase and amplitude in long-period seismograms, by normal-mode summation, without having to calculate complete splitting matrix elements (especially coupling terms), and without any ray tracing. This formulation is therefore well suited for inversion, as we shall show in a forthcoming paper.

This formulation is valid for large angular orders l and under the restriction that lateral heterogeneity is smooth ($s_{max} \ll 1$, if s_{max} is the maximum order of its spherical harmonics expansion). We note here that we have also assumed, in the course of our derivation, that we can, under these conditions, neglect perturbation terms arising from differences between the local frequencies and source radiation patterns of neighbouring multiplets of angular order l and $l+n$, with $|n| < s_{max}$. This conjecture remains to be proved, but is supported by the good agreement of our results with those obtained using a propagating wave approach in the high frequency limit where ray theory is applicable.

We have presented the derivation developed for spheroidal modes observed on the vertical component. It can be readily extended to horizontal components and toroidal modes in the same manner as was done for the higher order terms in the expression for the location parameter in Paper I. Also, the coupling effects due to the hydrostatic ellipticity of the earth can be calculated explicitly using known expressions for the corresponding elements of the splitting matrix (Woodhouse & Dahlen 1978), as well as for the rotation matrices for spherical harmonics (Edmonds 1960).

We have not considered here effects of coupling with the Earth's rotation, which concerns mainly low angular orders and has been addressed by other authors (e.g. Masters, Park & Gilbert 1983). Also, to obtain a complete asymptotic formulation for the first-order quasi degenerate splitting theory would require, in addition, inclusion of coupling terms between multiplets belonging to different dispersion branches that are close in frequency. These effects need to be attacked case by case and are beyond the scope of this study, as are effects due to anisotropy.

Finally, we have not considered here any possible effects due to anelasticity, such as lateral variations in the quality factor, which should be included in any inversion process.

Acknowledgments

The clarity of this paper has benefited greatly from critical reading by R. Madariaga and J. P. Montagner and a constructive review by R. Snieder. I also thank Jeff Park for pointing out an erroneous second-order term in the expression of the location parameter. This study was conducted under ASP 'Tomographie' grant no. 90 18 12 of the Institut National des Sciences de l'Univers. It is IPGP contribution no. 342.

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Appendix I

Woodhouse (1980) gives an expression for the general term of the splitting matrix $Z_{KK'} = H_{KK'} - \omega_k^2 P_{KK'}$ corresponding to the interaction of a mode K' with mode K, when the eigenfrequency $\omega_{k'}$ of mode K' in the reference spherically symmetric earth model is close to ω_k.

Let $\delta\mathbf{m} = \{\delta\kappa, \delta\mu, \delta\rho_0, \delta\phi_0, h\}$ be the vector of perturbations to the initial spherically symmetric elastic model of the earth, then the spherical harmonic expansion of the perturbation is:

$$\delta\mu = \delta\mu^{\mathrm{e}} + \sum_{st} \delta\mu_s^t Y_s^t \ldots, \tag{I.1}$$

where $\delta\mu^{\mathrm{e}}, \delta\kappa^{\mathrm{e}}, \ldots$, etc., are contributions from ellipticity. Then, setting aside contributions from ellipticity and rotation:

$$\begin{aligned} Z_{KK'}^{mm'} = \sum_{st} \Bigg(\int_0^a & (\delta\kappa_s^t K_s + \delta\mu_s^t M_s + \delta\rho_s^t R_s + \delta\phi_s^t G_s^1 + \delta\phi_s^t G_s^2)\, r^2\, dr \\ & - \sum_d r^2 h_s^t (\kappa \tilde{K}_s + \mu \tilde{M}_s + \rho \tilde{R}_s^1) \Bigg) (2l+1)^{1/2} (2s+1)^{1/2} (2l'+1)^{1/2} (4\pi)^{-1/2}\, W, \end{aligned} \tag{I.2}$$

Table 1. Expressions for the $i = 0$ kernels of the splitting matrix for: (a) a spheroidal mode K; (b) the coupling between two spheroidal modes K and K' (real part only).

(a) Single mode K	(b) Coupling terms
$M_{1K}^{(0)} = R_s = \frac{1}{2\omega_k}[8\pi G\rho_0 U^2 - g_0 U(F + 2r^{-1}U) + 2U\partial_r\phi_1 - \omega^2 U^2 - l(l+1)(\omega^2 V^2 - 2r^{-1}V\phi_1 - r^{-1}g_0 UV)]$	$\frac{1}{2\omega_{k'}}[\![8\pi G\rho_0 UU' - 1/2 g_0(4r^{-1}UU' + U'F + UF') - \omega_k^2 UU' + U'\partial_r\phi_1 + U\partial_r\phi_1' + 1/2[l(l+1) + l'(l'+1)]\,[-\omega_k^2 VV' + r^{-1}(\phi_1' V + \phi_1 V')] + 1/2 g_0 r^{-1}(U'V + V'U)]\!]$
$M_{2K}^{(0)} = M_s = \frac{1}{2\omega_k}[\![1/3(2\partial_r U - F)^2 + l(l+1)r^{-2}(r\partial_r V - V + U)^2 + l(l+1)\,[l(l+1) - 2]\,r^{-2}V^2]\!]$	$\frac{1}{2\omega_{k'}}[\![1/3(2\partial_r U - F)(2\partial_r U' - F') + r^{-2}(r\partial_r V - V + U)(r\partial_r V' - V' + U')(L + L') + r^{-2}\frac{VV'}{2}[(L' + L - 2)(L' + L) - 2LL']]\!]$
$M_{3K}^{(0)} = K_s = \frac{1}{2\omega_k}(\partial_r U + F)^2$	$\frac{1}{2\omega_{k'}}(\partial_r U + F)(\partial_r U' + F')$
$M_{4K}^{(0)} = G^{(1)} = 0$	$\frac{1}{2\omega_{k'}}1/2\rho_0 r^{-1}[\partial_r UV' - \partial_r U'V - 2FV' + 2F'V + r^{-1}(UV' - U'V)](L - L')$
$M_{5K} = G^{(2)} = \frac{1}{2\omega_k}\rho_0(-2UF)$	$\frac{1}{2\omega_{k'}}[-\rho_0(F'U + U'F) + 1/2\rho_0 r^{-1}(UV' - VU')(L - L')]$
$\tilde{K}_s = K_s - \frac{1}{2\omega_k}(\partial_r U + F)2\partial_r U$	$K_s - \frac{1}{2\omega_{k'}}[\![(2\partial_r U\partial_r U' + \partial_r U'F + \partial_r UF') + r^{-1}[(\partial_r U + F)V' - (\partial_r U' + F')V]\,(L - L')]\!]$
$\tilde{M}_s = M_s - \frac{1}{3\omega_k}(2\partial_r U - F)(2\partial_r U) - \frac{l(l+1)}{\omega_k}r^{-1}\partial_r V(rX)$	$M_s - \frac{1}{2\omega_{k'}}[\![r^{-1}(\partial_r V'X + \partial_r VX')(L - L') - 2/3[(2\partial_r U - F)\partial_r U' + (2\partial_r U' - F')\partial_r U]]\!] + \frac{1}{3\omega_{k'}r}[(2\partial_r U - F)V' - (2\partial_r U' - F')V]\,(L - L')$
With $F = r^{-1}[2U - l(l+1)V]$	Primes refer to mode K'
$X = V + r^{-1}(U - V)$	$L = l(l+1)$
	$L' = l'(l'+1)$

where W is the Wigner-2j symbol (Edmonds 1960):

$$W = \begin{pmatrix} l' & s & l \\ -m' & t & m \end{pmatrix}$$

with expressions for K_s, M_s ... as given in equations (A36)–(A42) of Woodhouse (1980). Following the procedure of Woodhouse & Girnius (1982), after some algebra, we can write $Z_{KK'}^{mm'}$ in the form:

$$Z_{KK'}^{mm'} = \sum_{st} \gamma_{st}^{kk'} \left(\int_0^a \delta m_s^t \, \| M_{KK'}^{(0)}(r) + s(s+1) M_{KK'}^{(1)}(r) + [s(s+1)^2 M_{KK}^{(2)}(r)] \, \| r^2 \, dr \right.$$
$$\left. - \sum_d h_s^t \, \| H_{KK'}^{(0)} + s(s+1) H_{KK'}^{(1)} + [s(s+1)]^2 H_{KK'}^{(2)} \|_-^+ \right), \tag{I.3}$$

with:

$$\gamma_{st}^{kk'} = \iint Y_l^{m*} \, Y_s^t Y_{l'}^{m'} \, d\Omega,$$

where $\mathbf{M}_{KK'}^{(i)}(r)$, $H_{KK'}^{(i)}(r)$ are summarized in Table 1 (for $i = 0$) and are given by known expressions in terms of scalar eigenfunctions of the reference model for modes K and K'.

Similarly to Woodhouse & Girnius (1982), this leads to defining three local functionals $\delta\omega_{kk'}^{(i)}$ of the Earth's structure, such that:

$$Z_{KK'}^{mm'} = \sum_{i=0}^{i=2} \iint \delta\omega_{kk'}^{(i)}(\theta, \phi) \, L_i^{mm'}(\theta, \phi) \, d\Omega, \tag{I.4}$$

where:

$$L_i^{mm'} = \frac{(-\nabla_1^2)^i Y_l^{m*} Y_{l'}^{m'}}{[2l(l+1)2l'(l'+1)]^{i/2}} \tag{I.5}$$

and

$$\delta\omega_{kk'}^{(i)}(\theta, \phi) = [2l(l+1)2l'(l'+1)]^{i/2} \left(\int_0^a \delta \mathbf{m}(r, \theta, \phi) \right) \mathbf{M}_{KK}^{(i)}(r) r^2 \, dr \tag{I.6}$$
$$- \sum_d h(\theta, \phi) \, (\mathbf{H}_{KK'}^{(i)})_-^+.$$

Table 1 also gives, for comparison, the corresponding expressions for the case of the splitting matrix elements $H_K^{mm'}$ for a single mode K as given by Woodhouse & Dahlen (1978) and summarized in table II of Woodhouse & Girnius (1982).

The expressions $M_{KK'}^{(i)}(r)$ contain imaginary parts, which, to order $1/l$, will yield zero contributions. Thus we do not consider them here.

Appendix II

APPROXIMATION OF INTEGRALS BY THE METHOD OF STATIONARY PHASE TO ORDER $1/l$

Let I be an integral of the form:

$$I(\lambda) = \int_A^B g(\lambda, \mu) \cos [kF(\mu)] \, d\mu, \tag{II.1}$$

where $k = l + 1/2$, and g is a slowly varying function of μ.

The contribution to this integral from the neighbourhood of a stationary point μ_0, such that $F'(\mu_0) = 0$, was derived in Paper I, Appendix I, and it is:

$$I_0 = \sqrt{\frac{2\pi}{k|F''|}}\left[\cos\left(kF_0 \pm \frac{\pi}{4}\right) g_0 - \frac{1}{k}\sin\left(kF_0 \pm \frac{\pi}{4}\right)\left(hg_0 + \frac{g_2}{2F''}\right) + 0\left(\frac{1}{l^2}\right)\right], \tag{II.2}$$

where

$$g_0 = g(\lambda, \mu_0)$$

$$F_0 = F(\mu_0)$$

$$F'' = \frac{\partial^2 F}{\partial \mu^2}(\mu_0)$$

$$g_2 = \frac{\partial^2 g}{\partial \mu^2}(\lambda, \mu_0). \tag{II.3}$$

The sign is plus or minus, according to the sign of F'', and h is a function of the 2nd, 3rd and 4th derivatives of F with respect to μ. In the case considered here, F is of the form:

$$kF(\mu) = k\beta - \frac{\pi}{4} + m\frac{\pi}{2} + \frac{n}{k}\cot\beta, \tag{II.4}$$

where m and n are integers and (Fig. 1):

$$\cos\beta = \cos\lambda\cos\Delta + \sin\lambda\sin\Delta\cos\mu. \tag{II.5}$$

Then:

$$h(\mu_0 = 0) = \frac{1}{8}\left(\frac{1}{\beta_0''} + 3\cot\beta_0\right), \tag{II.6}$$

where

$$\beta_0 = \beta(\lambda, 0); \quad \beta_0'' = \frac{\partial^2 \beta}{\partial \mu^2}(\lambda, 0). \tag{II.7}$$